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QUASITRIANGULAR + SMALL COMPACT = STRONGLY IRREDUCIBLE

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Abstract. Let T be a bounded linear operator acting on a separable infinite dimensional Hilbert space. Let ϵ be a positive number. In this article, we prove that the perturbation of T by a compact operator K with $\|K\| < \epsilon$ can be strongly irreducible if T is a quasitriangular operator with the spectrum $\sigma(T)$ connected. The Main Theorem of this article nearly answers the question below posed by D. A. Herrero.

Suppose that T is a bounded linear operator acting on a separable infinite dimensional Hilbert space with $\sigma(T)$ connected. Let $\epsilon > 0$ be given. Is there a compact operator K with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible?

1. Introduction

Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}$ be separable Hilbert spaces. Denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all bounded linear operators mapping $\mathcal{H}_1$ into $\mathcal{H}_2$. Denote by $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ the subset of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ of all compact operators. We simply write $\mathcal{B}(H)$ and $\mathcal{K}(H)$ instead of $\mathcal{B}(H, H)$ and $\mathcal{K}(H, H)$ respectively. For $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, denote the kernel of T and the range of T by Ker$T$ and Ran$T$ respectively. If $\mathcal{H}_0$ is a subspace of $\mathcal{H}$ (closed), we shall write $\mathcal{H}_0 \leq H$. Let $T \in \mathcal{B}(H)$; we shall denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_l(T)$, $\sigma_r(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, $\sigma_{lre}(T)$ and $\sigma_w(T)$ the spectrum, the point spectrum, the left spectrum, the right spectrum, the essential spectrum, the left essential spectrum, the Wolf spectrum and the Weyl spectrum of T respectively. Denote by $\sigma_0(T)$ the set of all isolated points of $\sigma(T)\backslash\sigma_e(T)$. For $\lambda \in \rho_{S-F}(T)$ ($\stackrel{\text{def}}{=} \mathcal{C}\backslash\sigma_{lre}(T)$), $\text{ind}(T-\lambda) = \text{dimKer}(T-\lambda) - \dim\text{Ker}(T-\lambda)^*$ and min $\text{ind}(T-\lambda) = min\{\text{dimKer}(T-\lambda), \dim\text{Ker}(T-\lambda)^*\}$. For $-\infty \leq n \leq +\infty$, $\rho^{(n)}_{S-F}(T) = \{\lambda \in \rho_{S-F}(T) : \text{ind}(T-\lambda) = n\}$. T is said to be *quasitriangular* if there is a sequence $\{P_n\}_{n\geq 1}$ of finite rank projections increasing to the unit operator I with respect to the strong operator topology such that $\lim_{n\to\infty} \|(I-P_n)TP_n\| = 0$. It is well-known that T is quasitriangular if and only if $\text{ind}(T-\lambda) \geq 0$ for all $\lambda \in \rho_{S-F}(T)$. T is said to be *strongly irreducible* if there are no nontrivial idempotents commuting with T. A *Cowen-Douglas operator* is an operator T satisfying the following conditions:

(i) There is a nonempty connected open subset Ω of $\rho^{(n)}_{S-F}(T)$ for a natural number n.

(ii) $T-\lambda$ is surjective for each $\lambda \in \Omega$.

(iii) $\bigvee\{\text{Ker}(T-\lambda) : \lambda \in \Omega\}$ is equal to the acting space of T.

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If the conditions above are satisfied, we shall write $T \in \mathcal{B}_n(\Omega)$. If $T \in \mathcal{B}_n(\Omega)$, then $\bigvee\{\mathrm{Ker}(T-\lambda)^k : k \geq 1\}$ is equal to the acting space of T for each λ in Ω.

Let σ be a compact subset of the complex field $\mathcal{C}$. A *clopen* σ_0 of σ is a subset of σ such that there are two disjoint open subsets Ω_1, Ω_2 of $\mathcal{C}$ such that $\Omega_1 \supset \sigma_0$ and $\Omega_2 \supset (\sigma \backslash \sigma_0)$.If σ is a clopen of $\sigma(T)$, then there is an analytic Cauchy domain Ω such that $\sigma(T) \cap \Omega = \sigma$ and such that $\sigma(T) \cap \partial\Omega = \emptyset$, where $\partial\Omega$ is the boundary of Ω. Thus $E(\sigma, T) = \frac{1}{2\pi i}\int_{\partial\Omega}(\lambda - T)^{-1}d\lambda$ is an idempotent commuting with T. We call $E(\sigma, T)$ the *Riesz idempotent* of T corresponding to σ. Write $\mathcal{H}(\sigma, T) = \mathrm{Ran}E(\sigma, T)$. It follows from the classical Riesz decomposition theorem that T is not strongly irreducible if $\sigma(T)$ is not connected. But the converse is not true. However, D. A. Herrero and C. L. Jiang obtained the approximate inverse of the Riesz decomposition theorem (see [3] or [6]):

Theorem HJ. *The closure of the class of all strongly irreducible operators is the class of all those operators which have connected spectrum.*

And then, D.A. Herrero posed the following question.

Question H. *Let T be an operator with $\sigma(T)$ connected. Given $\epsilon > 0$, can we find a compact operator K with $\| K\| < \epsilon$ such that $T + K$ is strongly irreducible?*

C.L. Jiang, S.H. Sun and Z.Y. Wang (see [10]) proved that if T is essentially normal and if $\sigma(T)$ is connected, then one can find a compact K such that $T + K$ is strongly irreducible. (But $\|K\|$ may be bigger than ϵ.) Y.Q. Ji, C.L. Jiang and Z.Y. Wang (see [8], [9]) proved that if T is an essentially normal quasitriangular operator with $\sigma(T)$ and $\sigma_\omega(T)$ connected, then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible. They (see [7]) also proved that if T is a Cowen-Douglas operator having unique (SI)-decomposition, then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible. C.L. Jiang, S. Power, and Z.Y. Wang (see [11]) proved that if T is a biquasitriangular operator with $\sigma(T)$ connected, then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.

The main result of this article is the theorem below.

Main Theorem. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasitriangular operator with $\sigma(T)$ connected and let $\varepsilon > 0$ be given. Then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.*

2. Preparation

In order to prove the Main Theorem, we need to prepare some lemmas.

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator with* $\mathrm{Ran}T$ *nonclosed. Then there is an infinite dimensional subspace $\mathcal{H}_0$ (closed) of $\mathcal{H}$ such that $\mathcal{H}_0 \cap \mathrm{Ran}T = \{0\}$.*

Proof. We know that $\mathrm{Ran}T = \mathrm{Ran}(TT^*)^{1/2}$. Without loss of generality, assume that T is positive and that $\mathrm{Ran}T$ is dense in $\mathcal{H}$. Let $E(*)$ be the spectral measure of T. It is easy to see that $E((0,t]) \neq 0$ for all $t > 0$ and that $E((0, \|T\|]) = I$. Choose a sequence $\{t_k\}_{k\geq 0}$ of positive numbers decreasing to zero such that $t_0 = \|T\|$ and such that $E((t_k, t_{k-1}]) \neq 0$ for all $k \geq 1$. Write $E_k = E((t_k, t_{k-1}])$. Let $\mathcal{H}_n = \bigvee\{\mathrm{Ran}E_{(2k-1)2^{n-1}} : k \geq 1\}$. Then $\{\mathcal{H}_n\}_{n\geq 1}$ is a pairwise orthogonal family of subspaces and $\mathcal{H} = \bigoplus\limits_{n\geq 1} \mathcal{H}_n$. Let P_n be the projection onto $\mathcal{H}_n$, i.e.

$P_n = \sum_{k\geq 1} E_{(2k-1)2^{n-1}}$. It follows that $P_nT = TP_n$. It is not difficult to show that $0 \in \sigma(P_nT|_{\mathcal{H}_n})$. Hence we can take $x_n \in \mathcal{H}_n \backslash \mathrm{Ran}(P_nT|_{\mathcal{H}_n})$ for each $n \geq 1$. Let $\mathcal{H}_0 = \bigvee\{x_n, n \geq 1\}$. Then $\dim \mathcal{H}_0 = +\infty$ and $\mathcal{H}_0 \cap \mathrm{Ran}T = \{0\}$. In fact, if $Ty = x = \sum_{n\geq 1} \alpha_n x_n$, then $\alpha_n x_n = P_n x = P_n Ty = P_nTP_n y$. So $\alpha_n = 0$. And then $x = 0$. □

Remark. If there is an infinite dimensional linear submanifold $\mathcal{X}$ of $\mathcal{H}$ such that $\mathcal{X}$ $\cap \mathrm{Ran}T = \{0\}$, then it follows from Lemma 2.1 that there is an infinite dimensional subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $\mathcal{H}_0 \cap \mathrm{Ran}T = \{0\}$.

Lemma 2.2. *Let* $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$. *Suppose that* $\mathrm{Ran}B \subset \mathrm{Ran}A$ *and suppose that* $\mathcal{X}$ *is an infinite dimensional linear submainfold of* $\mathcal{H}$ *such that* $\mathcal{X} \cap \mathrm{Ran}A = \{0\}$. *Given* $\epsilon > 0$, *then there exists a compact operator* $K \in \mathcal{K}(\mathcal{H}_2, \mathcal{H})$ *with* $\|K\| < \epsilon$ *such that* $\mathrm{Ran}A \cap \mathrm{Ran}(B+K) = \{0\}$ *and such that* $\mathrm{Ker}(B+K) = \{0\}$.

Proof. By the remark above, find $\mathcal{H}_0 < \mathcal{H}$ such that $\mathcal{H}_0 \cap \mathrm{Ran}A = \{0\}$ and $\dim \mathcal{H}_0 = \infty$. Take an injective $K \in \mathcal{K}(\mathcal{H}_2, \mathcal{H})$ mapping $\mathcal{H}_2$ into $\mathcal{H}_0$ and with $\|K\| < \epsilon$. If $u = Ax = (B + K)y = By + Ky$, it follows by $\mathrm{Ran}B \subset \mathrm{Ran}A$ that $Ky \in \mathrm{Ran}A \cap \mathcal{H}_0 = \{0\}$. So $Ky = 0$. Since K is injective, $y = 0$. Hence $\mathrm{Ran}A \cap \mathrm{Ran}(B + K) = \{0\}$. Similarly, $\mathrm{Ker}(B + K) = \{0\}$. □

Lemma 2.3 ([7]). *Let* $T \in \mathcal{B}(\mathcal{H})$. *Suppose that* $\dim \mathrm{Ker}\, T = 1$ $\dim \mathrm{Ker}\, T = 1$ *and suppose that* $\bigvee_{n\geq 1} \mathrm{Ker}T^n = \mathcal{H}$. *Then* T *is strongly irreducible.*

Lemma 2.4. *Set*

$$T = \begin{bmatrix} T_1 & T_{12} \\ & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

where the entry omitted is 0. *Suppose that*

(i) $\mathrm{dimKer}T_1 = 1$, $\bigvee_{n\geq 1} \mathrm{Ker}T_1^n = \mathcal{H}_1$,

(ii) $\bigvee_{n\geq 1} \mathrm{Ker}T_2^n = \mathcal{H}_2$,

(iii) $\mathrm{Ker}T_{12} \cap \mathrm{Ker}T_2 = \{0\}$,

(iv) $\mathrm{Ran}T_1 \cap \mathrm{Ran}(T_{12} \mid_{\mathrm{Ker}T_2}) = \{0\}$. ($\mathrm{Ran}(T_{12} \mid_{\mathrm{Ker}T_2}) = T_{12}(\mathrm{Ker}T_2)$.)

Then T *is strongly irreducible.*

Proof. Suppose $T(x \oplus y) = 0$, where $x \in \mathcal{H}_1, y \in \mathcal{H}_2$. Computation shows that $T_2 y = 0$ and $T_1 x + T_{12} y = 0$. By (iv), $T_1 x = 0$, i.e. $x \in \mathrm{Ker}T_1$, and $T_{12} y = 0$. It follows from (iii) that $y = 0$. So $\mathrm{Ker}T = \mathrm{Ker}T_1$. It is easy to show that $\bigvee_{n\geq 1} \mathrm{Ker}T^n = \mathcal{H}_1$. Suppose that P is an idempotent commuting with T. Then $PT^n = T^n P$ for $n \geq 1$. So $P(\mathrm{Ker}T^n) \subset \mathrm{Ker}T^n$ for all $n \geq 1$. Thus $\mathcal{H}_1 \in \mathrm{Lat}\, P$. Set

$$P = \begin{bmatrix} P_1 & P_{12} \\ & P_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

Then $P_i^2 = P_i$ and $P_i T_i = T_i P_i$, $i = 1, 2$. By Lemma 2.3 and the condition (i), $P_1 = I|_{\mathcal{H}_1}$ or 0. Assume $P_1 = 0$ (otherwise, consider $I - P$). Computing the (1,2)-entry shows that $P_{12} T_2 = T_1 P_{12} + T_{12} P_2$. Let $y \in \mathrm{Ker}T_2$. It follows that $P_2 y \in \mathrm{Ker}T_2$, and so $P_2 T_2 = T_2 P_2$. So $T_1 P_{12} y = -T_{12} P_2 y \in \mathrm{Ran}T_1 \cap$ $\mathrm{Ran}(T_{12} \mid_{\mathrm{Ker}T_2})$. By the condition (iv), $P_2 y = 0$. Hence $P_2(\mathrm{Ker}T_2) = \{0\}$. If $x \in \mathrm{Ker}T_2^2$, then $\mathrm{T}_2 x \in \mathrm{Ker}T_2$.

This shows that $T_2P_2x = P_2T_2x = 0$, $P_2x \in \mathrm{Ker}T_2$. Thus $P_2x = P_2(P_2x) = 0$. So $P_2(\mathrm{Ker}T_2^2) = \{0\}$. Inductively, $P_2(\mathrm{Ker}T_2^n) = \{0\}$ for all $n \geq 1$. By the condition (ii), $P_2 = 0$. So $P = P^2 = 0$, and T is strongly irreducible. ☐

Lemma 2.5. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ and that T satisfies the following conditions:*

(i) $0 \in \partial\sigma(T)$ *(the boundary of $\sigma(T)$),*

(ii) $\mathrm{Ker}\ T \subset \bigcap_{n\geq 1} \mathrm{Ran}T^n$,

(iii) $\bigvee_{n\geq 1} \mathrm{Ker}T^n = \mathcal{H}$.

Let $\epsilon > 0$ be given. Then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.

Proof. By Lemma 2.3, we only need to show this lemma in the case that $\dim\mathrm{Ker}T > 1$. Choose $x_0 \in \mathrm{Ker}T\backslash\{0\}$. Let $\mathcal{N}_\infty = \mathrm{Ker}T \ominus \mathcal{C}x_0$. By the condition (ii), we can take $x_k \in \mathcal{N}_\infty^\perp$ such that $Tx_k = x_{k-1}$ for each $k \geq 1$. Let $\mathcal{H}_1 = \bigvee\{x_k : 1 \leq k < +\infty\}$. Then $\mathcal{H}_1 \in \mathrm{Lat}\, T$ and $\mathcal{N}_\infty \subset \mathcal{H}_1^\perp$. Set

$$T = \begin{bmatrix} T_1 & T_{12} \\ & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 = \mathcal{H}_1^\perp \end{matrix} \tag{1}$$

It is easy to see that the following hold.

(1) $\dim\mathrm{Ker}T_1 = 1$, $\bigvee_{n\geq 1} \mathrm{Ker}T_1^n = \mathcal{H}_1$ and $\overline{\mathrm{Ran}T^1} = \mathcal{H}_1$.

(2) $0 \in \partial\sigma(T_1)$ (this follows from $0 \in \partial\sigma(T)$).

(3) $\bigvee_{n\geq 1} \mathrm{Ker}T_2^n = \mathcal{H}_2$ (It follows from that $\bigcap_{n\geq 1} \overline{\mathrm{Ran}T_2^{*n}} \subset \bigcap_{n\geq 1} \overline{\mathrm{Ran}T^{*n}} = \{0\}$).

(4) $\mathcal{N}_\infty \subset \mathrm{Ker}T_2$ and $T_{12}(\mathcal{N}_\infty) = \{0\}$

(5) $\mathrm{Ker}(T_{12}|_{\mathrm{Ker}T_2 \ominus \mathcal{N}_\infty}) = \{0\}$ and $\mathrm{Ran}T_1 \cap T_{12}(\mathrm{Ker}T_2 \ominus \mathcal{N}_\infty) = \{0\}$.

Let A be an operator mapping $(\mathcal{H}_1 \ominus \mathrm{Ker}T_1) \oplus (\mathrm{Ker}T_2 \ominus \mathcal{N}_\infty)$ into $\mathcal{H}_1$ such that $A(x \oplus y) = T_1x + T_{12}y$. By (5) above, $\mathrm{Ker}A = \{0\}$. Since $0 \in \partial\sigma(T_1)$ and $\overline{\mathrm{Ran}T^1} = \mathcal{H}_1$, $A|_{\mathcal{H}_1 \ominus \mathrm{Ker}T_1}$ is unbounded from below. So $\mathrm{Ran}A$ is nonclosed. By Lemma 2.2, we can take a $B \in \mathcal{K}(\mathcal{N}_\infty, \mathcal{H}_1)$ with $\|B\| < \epsilon$ and $\mathrm{Ker}B = \{0\}$ such that $\mathrm{Ran}B \cap \mathrm{Ran}A = \{0\}$. Define

$$Kx = \begin{cases} Bx, & x \in \mathcal{N}_\infty, \\ 0, & x \in \mathcal{N}_\infty^\perp. \end{cases}$$

Then $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \epsilon$. It is easy to see that

$$T + K = \begin{bmatrix} T_1 & C \\ & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} \tag{2}$$

satisfies $\mathrm{Ker}C \cap \mathrm{Ker}T_2 = \{0\}$ and $\mathrm{Ran}T_1 \cap C(\mathrm{Ker}T_2) = \{0\}$. By Lemma 2.4, $T + K$ is strongly irreducible. ☐

Lemma 2.6. *Let T be an operator acting on $\mathcal{H}$ satisfying the following conditions:*

(i) $0 \in \partial\sigma(T)$ *and* $\bigvee_{n\geq 1} \mathrm{Ker}T^n = \mathcal{H}$.

(ii) $\mathrm{Ker}T \cap (\bigcap_{n\geq 1} \mathrm{Ran}T^n)$ *is closed and* $\dim\mathrm{Ker}T \ominus (\mathrm{Ker}T \cap (\bigcap_{n\geq 1} \mathrm{Ran}T)) < \infty$.

Let $\epsilon > 0$ be given. Then there exists a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.

Proof. Write $\mathcal{N}_\infty = \mathrm{Ker}T \cap (\bigcap_{n\geq 1} \mathrm{Ran}T^n)$. Denote $\mathrm{Ker}T \ominus \mathcal{N}_\infty = \mathcal{N}_0$. By the condition (ii), $\dim\mathcal{N}_0 < +\infty$. For $k \geq 1$, we can inductively define $\mathcal{N}_k = \{x :$

$Tx \in \mathcal{N}_{k-1}, x \perp \mathcal{N}_\infty\}$. Since $\dim \mathcal{N}_0 < +\infty$, $\mathcal{N}_0 \cap \mathrm{Ran} T^{n_0} = \{0\}$ for some n_0. So $\mathcal{N}_k = \mathcal{N}_0$ when $n_0 \le k < +\infty$. Thus $\bigvee\{\mathcal{N}_k : k < +\infty\} = \mathcal{N}_{n_0}$. Denote it by $\mathcal{H}_1$. Then $\mathcal{H}_1 \in \mathrm{Lat}\, T$ and $\dim \mathcal{H}_1 (\stackrel{\text{def}}{=} m) < +\infty$. Set

$$T = \begin{bmatrix} T_1 & T_{12} \\ & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 = \mathcal{H}_1^\perp \end{matrix}$$

It is not difficult to show that

(1) $\mathrm{Ker} T_2 = \mathcal{N}_\infty \subset \bigcap_{n\ge 1} \mathrm{Ran} T_2^n$.

(2) $\bigvee_{n \ge 1} \mathrm{Ker} T_2^n = \mathcal{H}_2$.

(3) $T_1^m = 0$.

Choose $C \in \mathcal{K}(\mathcal{H}_1)$ with $\|C\| < \frac{\epsilon}{2}$ $(T_1+C)^{m-1} \neq 0$ and such that $(T_1+C)^m = 0$. Take unit vectors $f \in \mathcal{H}_1 \ominus \mathrm{Ran}(T_1 + C)$ and $e \in \mathcal{N}_\infty$. Set

$$K_1 = \begin{bmatrix} C & \frac{\epsilon}{4} f \otimes e \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

where $(f \otimes e)x = (x, e)f$. Then $K_1 \in \mathcal{K}(\mathcal{H})$ and $\|K_1\| < \frac{3\epsilon}{4}$. It is not difficult to show that $\mathrm{Ker}(T + K_1) = \mathrm{Ker}(T_1 + C) \oplus (\mathcal{N}_\infty \ominus \mathcal{C}e) \subset \bigcap_{n\ge 1} \mathrm{Ran}(T + K_1)^n$. It is clear that $0 \in \partial\sigma(T + K_1)$. By Lemma 2.5, one can find a $K_2 \in \mathcal{K}(\mathcal{H})$ with $\|K_2\| < \frac{\epsilon}{4}$ such that $T + K_1 + K_2$ is strongly irreducible. Let $K = K_1 + K_2 \in \mathcal{K}(\mathcal{H})$. Then $\|K\| < \epsilon$. □

Let $T \in \mathcal{K}(\mathcal{H})$ have the following form:

$$T = \begin{bmatrix} 0 & A_1 & * & * & \cdots \\ & 0 & A_2 & * & \cdots \\ & & 0 & A_3 & \cdots \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} \begin{matrix} \mathrm{Ker} T \\ \mathrm{Ker} T^2 \ominus \mathrm{Ker} T \\ \mathrm{Ker} T^3 \ominus \mathrm{Ker} T^2 \\ \mathrm{Ker} T^4 \ominus \mathrm{Ker} T^3 \\ \vdots \end{matrix} \tag{3}$$

It is easy to see that $\mathrm{Ker} A_i = \{0\}$ and $\mathrm{Ker} T \cap\ \mathrm{Ran} T^i = \mathrm{Ran}(A_1 A_2 \cdots A_i)$ for all $i \ge 1$. Thus $\mathrm{Ker} T \cap (\bigcap_{n\ge 1} \mathrm{Ran} T^n) = \bigcap_{n \ge 1} \mathrm{Ran}(A_1 A_2 \cdots A_i)$. It follows that $\mathrm{Ker} T \cap (\bigcap_{n\ge 1} \mathrm{Ran} T^n)$ is closed when $\mathrm{Ran} A_n$ is closed for each $n \ge 1$.

Lemma 2.7. *Let T be as above. Suppose that n_0 is a natural number and suppose that $\mathcal{M}$ is an infinite dimensional subspace of $\mathrm{Ker} T^{n_0} \ominus \mathrm{Ker} T^{n_0 - 1}$ such that $\mathcal{M} \cap \mathrm{Ran} A_{n_0} = \{0\}$. Let $\epsilon > 0$ be given. Then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.*

Proof. Without loss of generality, assume that $n_0 > 1$ and that A_i has closed range with finite codimension for each $i < n_0$. Let $T_1 = T|_{\mathrm{Ker} T^{n_0-1}}$, i.e.

$$T_1 = \begin{bmatrix} 0 & A_1 & \cdots & * & * \\ & 0 & \ddots & * & * \\ & & \ddots & & \vdots \\ & & & 0 & A_{n_0-2} \\ & & & & 0 \end{bmatrix} \begin{matrix} \mathrm{Ker} T \\ \mathrm{Ker} T^2 \ominus \mathrm{Ker} T \\ \vdots \\ \mathrm{Ker} T^{n_0-2} \ominus \mathrm{Ker} T^{n_0-3} \\ \mathrm{Ker} T^{n_0-1} \ominus \mathrm{Ker} T^{n_0-2} \end{matrix}$$

Since Ran A_i is closed for each $i < n_0$, T_1 is similar to $\bigoplus_{j=1}^{\infty} J_j$, where J_j is a Jordon block for each j. So we can find $C_1 \in \mathcal{K}(\mathrm{Ker}T^{n_0-1})$ with $\|C_1\| < \frac{\epsilon}{2}$ such that $\mathrm{dimKer}(T_1 + C_1) = 1$ and $\bigvee_{n\geq 1} \mathrm{Ker}(T_1 + C_1)^n = \mathrm{Ker}T^{n_0-1}$. It is clear that

$$\mathrm{Ran}(T_1 + C_1) \neq \overline{\mathrm{Ran}(T_1 + C_1)} = \mathrm{Ker}T^{n_0-1}.$$

Let P be the projection onto $(\mathrm{Ker}T^{n_0-1})^{\perp}$. Write $T_2 = PT|_{\mathrm{Ran}P}$. Then

$$T_2 = \begin{bmatrix} 0 & A_{n_0} & * & \cdots \\ & 0 & A_{n_0+1} & \cdots \\ & & 0 & \\ & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{N}_1 = \mathrm{Ker}T^{n_0} \ominus \mathrm{Ker}T^{n_0-1} \\ \mathcal{N}_2 = \mathrm{Ker}T^{n_0+1} \ominus \mathrm{Ker}T^{n_0} \\ \mathcal{N}_3 = \mathrm{Ker}T^{n_0+2} \ominus \mathrm{Ker}T^{n_0+1} \\ \vdots \end{matrix}$$

Decompose $\mathcal{N}_1$ as $\mathcal{M} \oplus (\mathcal{N}_1 \ominus \mathcal{M})$. Then

$$T_2 = \begin{bmatrix} 0 & 0 & B_2 & * & \cdots \\ & 0 & B_1 & * & \cdots \\ & & 0 & A_{n_0+1} & \cdots \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{N}_1 \ominus \mathcal{M} \\ \mathcal{N}_2 \\ \mathcal{N}_3 \\ \vdots \end{matrix}$$

where $\begin{bmatrix} B_2 \\ B_1 \end{bmatrix} = A_{n_0}$. For each $0 \neq x \in \mathcal{N}_2$, $A_{n_0}x = B_2x + B_1x \notin \mathcal{M}$. So $B_1x \neq 0$. Set

$$T_3 = \begin{bmatrix} 0 & B_1 & * & \cdots \\ & 0 & A_{n_0+1} & \cdots \\ & & 0 & \\ & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{N}_1 \ominus \mathcal{M} \\ \mathcal{N}_2 \\ \mathcal{N}_3 \\ \vdots \end{matrix}$$

Then $\mathrm{Ker}T_3 = \mathcal{N}_1 \ominus \mathcal{M}$, and T can be written as

$$(4) \qquad T = \begin{bmatrix} T_1 & T_{12} & * \\ & 0 & T_{23} \\ & & T_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 = \mathrm{Ker}T^{n_0-1} \\ \mathcal{M} \\ \mathcal{H}_2 = \mathcal{H} \ominus (\mathcal{H}_1 \oplus \mathcal{M}) \end{matrix}$$

where $T_{23}|_{\mathrm{Ker}T_3} = 0$. Take $C_2 \in \mathcal{K}(\mathcal{M})$ with $\|C_2\| < \frac{\epsilon}{4}$ such that $\mathrm{dimKer}C_2 = 1$ and such that $\bigvee_{n\geq 1} \mathrm{Ker}C_2^n = \mathcal{M}$. By Lemma 2.2, choose $C_3 \in \mathcal{K}(\mathcal{H}_2, \mathcal{M})$ with $\|C_3\| < \frac{\epsilon}{8}$ such that $\mathrm{Ran}C_3 \cap \mathrm{Ran}C_2 = \{0\}$ and such that $\mathrm{Ker}C_3 = \mathcal{H}_2 \ominus \mathrm{Ker}T_3$. Take a rank one operator $C_4 \in \mathcal{K}(\mathcal{M}, \mathcal{H}_1)$ with $\|C_4\| < \frac{\epsilon}{16}$ such that $\mathrm{Ker}C_4 = \mathcal{M} \ominus \mathrm{Ker}C_2$, $\mathrm{Ker}(T_{12} + C_4) \cap \mathrm{Ker}C_2 = \{0\}$ and $(T_{12} + C_4)(\mathrm{Ker}C_2) \cap \mathrm{Ran}(T_1 + C_1) = \{0\}$. Set

$$K = \begin{bmatrix} C_1 & C_4 & 0 \\ & C_2 & C_3 \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{M} \\ \mathcal{H}_2 \end{matrix}$$

Then $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \epsilon$, and so

$$(5) \qquad T + K = \begin{bmatrix} T_1 + C_1 & T_{12} + C_4 & * \\ & C_2 & T_{23} + C_3 \\ & & T_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{M} \\ \mathcal{H}_2 \end{matrix}$$

Similarly to the proof of Lemma 2.4, one can with no difficulty verify that $T + K$ is strongly irreducible. □

Lemma 2.8. *Suppose that T has the form (3) and that the following conditions are satisfied:*

(i) $\overline{\text{Ran}A_i} = \text{Ran}A_i$ *and* $\dim\text{Ker}A_i^* < +\infty$ *for each* i.

(ii) $\dim\text{Ker}T \cap \bigcap_{n\geq 1} (\text{Ran}T^n) = +\infty$.

(iii) $\dim\text{Ker}T \ominus (\text{Ker}T \cap (\bigcap_{n\geq 1} \text{Ran}T^n)) = +\infty$.

Let $\epsilon > 0$ be given. Then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.

Proof. Write $\mathcal{N}_\infty = \text{Ker}T \cap (\bigcap_{n\geq 1} \text{Ran}T^n)$ and $\mathcal{N}_0 = \text{Ker}T \ominus \mathcal{N}_\infty$. For $1 \leq k < +\infty$, inductively define $\mathcal{N}_k = \{x : Tx \in \mathcal{N}_{k-1}, x \perp \mathcal{N}_\infty\}$, and set $\bigvee\{\mathcal{N}_k : 0 \leq k < +\infty\} = \mathcal{H}_1$. By (i) and (iii), there is an infinite dimensional linear submanifold $\mathcal{X}$ of $\mathcal{H}_1$ such that $\mathcal{X} \cap \text{Ran}T = \{0\}$. It is clear that $\mathcal{H}_1 \in \text{Lat}\,T$. Set

$$T = \begin{bmatrix} T_1 & T_{12} \\ & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1^\perp \end{matrix}$$

It is easy to see that

(1) $\text{Ker}T_1 = \mathcal{N}_0$ and $\bigvee_{n\geq 1} \text{Ker}T_1^n = \mathcal{H}_1$,

(2) $\mathcal{N}_\infty \subset \text{Ker}T_2$ and $\bigvee_{n\geq 1} \text{Ker}T_2^n = \mathcal{H}_1^\perp$.

Write $\mathcal{M} = A_1^{-1}(\mathcal{N}_\infty)$. Since $\text{Ran}A_1$ is closed and $\text{Ker}A_1 = \{0\}$, we can find a positive number r such that $r\,\|x\| \leq \|A_1 x\|$ for $x \in \mathcal{M}$. Write $\mathcal{L} = P_{\mathcal{H}_1^\perp}\mathcal{M}$, where $P_{\mathcal{H}_1^\perp}$ is the projection onto $\mathcal{H}_1^\perp$. Suppose $x = x_1 \oplus x_2 \in \mathcal{M}$, $x_1 \in \mathcal{H}_1$, $x_2 \in \mathcal{L}$.

$$\|P_{\mathcal{H}_1^\perp} x\| = \|x_2\| \geq \frac{\|T_2 x_2\|}{\|T_2\|} = \frac{\|A_1 x_2\|}{\|T_2\|} \geq \frac{r}{\|T_2\|}\|x\|.$$

So $\mathcal{L}$ is closed. Moreover, $T_{12}y \in \text{Ran}T_1$ for all $y \in \mathcal{L}$. Write $\mathcal{H}_2 = \mathcal{H}_1^\perp \ominus \mathcal{N}_\infty$. Set

$$T_2 = \begin{bmatrix} 0 & T_{23} \\ & T_3 \end{bmatrix} \begin{matrix} \mathcal{N}_\infty \\ \mathcal{H}_2 \end{matrix}$$

It is easy to see that $\text{Ker}T_3 = \mathcal{L} \oplus (\text{Ker}T_2 \ominus \mathcal{N}_\infty)$ and that $\bigvee_{n\geq 1} \text{Ker}T_3^n = \mathcal{H}_2$. If $0 \neq y \in \text{Ker}T_3 \ominus \mathcal{L}$, then $T_{12}y \notin \text{Ran}T_1$. Notice that $\mathcal{X} \cap (\text{Ran}T_1 + T_{12}(\text{Ker}T_3 \ominus \mathcal{L})) \subset \mathcal{X} \cap \text{Ran}T = \{0\}$ and that

$$T = \begin{bmatrix} T_1 & 0 & T_{12}|_{\mathcal{H}_2} \\ & 0 & T_{23} \\ & & T_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{N}_\infty \\ \mathcal{H}_2 \end{matrix} = \begin{bmatrix} 0 & 0 & T_{23} \\ & T_1 & T_{12}|_{\mathcal{H}_2} \\ & & T_3 \end{bmatrix} \begin{matrix} \mathcal{N}_\infty \\ \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} \tag{6}$$

Similarly to the proof of Lemma 2.5, by Lemma 2.2 there is a $C \in \mathcal{K}(\mathcal{H}_2, \mathcal{H}_1)$ with $\|C\| < \frac{\epsilon}{2}$ such that $\text{Ker}C = \mathcal{H}_2 \ominus \mathcal{L}$ and $\text{Ran}C \cap (\text{Ran}T_1 + T_{12}(\text{Ker}T_3 \ominus \mathcal{L})) = \{0\}$. Write $B = C + T_{12}|_{\mathcal{H}_2}$. Then $\text{Ran}T_1 \cap B(\text{Ker}T_3) = \{0\}$. Take $T_0 \in \mathcal{K}(\mathcal{N}_\infty)$ with $\|T_0\| < \frac{\epsilon}{4}$ such that $\dim\text{Ker}T_0 = 1$ and such that $\bigvee_{n\geq 1} \text{Ker}T_0^n = \mathcal{N}_\infty$. Since $\dim\mathcal{N}_\infty = +\infty$, $\text{Ran}T_0 \neq \overline{\text{Ran}T_0} = \mathcal{N}_\infty$. By Lemma 2.2, we can find a $D \in \mathcal{K}(\mathcal{H}_1, \mathcal{N}_\infty)$ with $\|D\| < \frac{\epsilon}{8}$ such that $\text{Ran}D \cap \text{Ran}T_0 = \{0\}$ and $\text{Ker}D = \mathcal{H}_1 \ominus \text{Ker}T_1$. Set

$$K = \begin{bmatrix} T_0 & D & 0 \\ & 0 & C \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{N}_\infty \\ \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

Then $K \in \mathcal{K}(\mathcal{H})$, $\|K\| < \epsilon$ and

$$T + K = \begin{bmatrix} T_0 & D & * \\ & T_1 & B \\ & & T_3 \end{bmatrix} \begin{matrix} \mathcal{N}_\infty \\ \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} \tag{7}$$

Similarly to the proof of Lemma 2.4, one can show that $T+K$ is strongly irreducible. □

By the equivalence of (str-v)$_{-m}$ and (str-vi)$_{-m}$ of Theorem 1.2 in [4], we have the following lemma.

Lemma 2.9 ([4])**.** *Let m be a natural number and let $T \in \mathcal{B}(\mathcal{H})$ be a quasitriangular operator with $\sigma(T)$ and $\sigma_w(T)$ connected. Let $\epsilon > 0$ and $\lambda \in \sigma_e(T) \cup (\bigcup_{k \geq m} \rho^{(k)}_{S-F}(T))$ be given. Then there exist a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ and a sequence $\{P_n\}_{n\geq 0}$ of finite rank projections increasing to I with respect to the strong operator topology with* $\text{rank} P_n = mn$ *such that $(I - P_n)(T - \lambda + K)P_n = 0$ for all $n \geq 0$, i.e.*

$$T + K - \lambda = \begin{bmatrix} 0 & * & * & * & \cdots \\ & 0 & * & * & \cdots \\ & & 0 & * & \cdots \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} \begin{matrix} \text{Ran}P_1 \\ \text{Ran}(P_2 - P_1) \\ \text{Ran}(P_3 - P_2) \\ \text{Ran}(P4 - P_3) \\ \vdots \end{matrix}$$

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasitriangular operator with $\sigma(T)$ and $\sigma_w(T)$ connected. Let $\epsilon > 0$ be given. Then there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K$ is strongly irreducible.*

Proof. Without loss of generality, assume that $0 \in \partial\sigma(T)$. By Lemma 2.9, find $K_1 \in \mathcal{K}(\mathcal{H})$ with $\|K_1\| < \frac{\epsilon}{4}$ and a sequence $\{P^{(1)}_n\}_{n\geq 0}$ of finite rank projections increasing to I with respect to the strong operator topology so that

$$T + K_1 = \begin{bmatrix} 0 & * & * & \cdots \\ & 0 & * & \cdots \\ & & 0 & \\ & & & \ddots \end{bmatrix} \begin{matrix} \text{Ran}P^{(1)}_1 \\ \text{Ran}(P^{(1)}_2 - P^{(1)}_1) \\ \text{Ran}(P^{(1)}_3 - P^{(1)}_2) \\ \vdots \end{matrix}$$

It is obvious that $\sigma_p((T + K_1)^*) \subset \{0\}$. While $0 \in \sigma_{lre}(T)$,

$$\sigma(T + K_1) = \sigma_w(T + K_1) = \sigma_w(T).$$

Write $P_1 = P^{(1)}_1$ and $\mathcal{N}_1 = \text{Ran}P^{(1)}_1$. Let $T_1 = (I - P_1)(T + K_1)|_{\text{Ran}(I-P_1)}$. It is not difficult to show that $\sigma(T_1) = \sigma_w(T_1) = \sigma_w(T)$, $\sigma_{lre}(T_1) = \sigma_{lre}(T)$ and $\text{ind}(T_1 - \lambda) = \text{ind}(T - \lambda)$ for all $\lambda \in \rho_{S-F}(T_1)$. Applying Lemma 2.9 to T_1, one can find a compact operator $K^{(2)} \in \mathcal{K}(\mathcal{H})$ with $\|K^{(2)}\| < \frac{\epsilon}{8}$ and a sequence $\{P^{(2)}_n\}_{n\geq 0}$ of finite rank projections increasing to $I|_{\text{Ran}(I-P_1)}$ with respect to the strong operator topology such that $\text{rank}P_1^{(2)} = 2\text{rank}P_1$ and

$$T_1 + K^{(2)} = \begin{bmatrix} 0 & * & * & \cdots \\ & 0 & * & \cdots \\ & & 0 & \\ & & & \ddots \end{bmatrix} \begin{matrix} \text{Ran}P_1^{(2)} \\ \text{Ran}(P_2^{(2)} - P_1^{(2)}) \\ \vdots \\ \vdots \end{matrix}$$

Since $\text{rank}P_2^{(1)} < +\infty$, there is a natural number n_1 such that $\|(I - P_2)P_2^{(1)}\| < \frac{1}{2}$, where P_2 is the projection onto $\text{Ran}P_1 \oplus \text{Ran}P_{n_1}^{(2)}$. Write $\mathcal{N}_2 = \text{Ran}P_1^{(2)}$ and $\mathcal{N}_j = \text{Ran}(P_{j-1}^{(2)} - P_{j-2}^{(2)})$ for $2 < j \le n_1 + 1$. Set

$$K_2 = \begin{bmatrix} 0 & \\ & K^{(2)} \end{bmatrix} \begin{matrix} \text{Ran}P_1 \\ \text{Ran}(I - P_1) \end{matrix}$$

Then $K_2 \in \mathcal{K}(\mathcal{H})$ and $\|K_2\| < \frac{\epsilon}{8}$. Let $T_2 = (I-P_2)(T+K_1+K_2)|_{\text{Ran}(I-P_2)}$. One can show that $\sigma(T_2) = \sigma_w(T_2) = \sigma_w(T)$, $\sigma_{lre}(T_2) = \sigma_{lre}(T)$ and $\text{ind}(T_2-\lambda) = \text{ind}(T-\lambda)$ for all $\lambda \in \rho_{S-F}(T_2)$.

Repeatedly using the process above, we can inductively choose a sequence $\{n_i\}_{i\ge1}$ of natural numbers, a sequence $\{\mathcal{N}_k\}_{k\ge1}$ of pairwise orthogonal finite dimensional subspaces, an increasing sequence $\{P_k\}$ of finite rank projections and a sequence $\{K_n\}_{n\ge1}$ of compact operators such that

(i) $\dim\mathcal{N}_k \le \dim\mathcal{N}_{k+1} < +\infty$ $(k \ge 1)$,

(ii) $\text{Ran}P_k = \bigoplus\{\mathcal{N}_j : j \le 1 + \sum_{i=1}^{k-1} n_i\}$ $(k \ge 1)$,

(iii) $\dim\mathcal{N}_{2+\sum_{i=1}^{k-1} n_i} = 2^k \,\text{rank}P_k$ $(k \ge 1)$,

(iv) $\|(I - P_n)P_n^{(1)}\| < \frac{1}{n}$ for $n \ge 1$, and hence $\bigoplus_{1\le k<+\infty} \mathcal{N}_k = \mathcal{H}$.

(v) $\|K_n\| < \frac{\epsilon}{2^{n+1}}$, hence $\overline{K}_1 = \sum_{1\le k<+\infty} K_n \in \mathcal{K}(\mathcal{H})$ and $\|\overline{K}_1\| < \frac{\epsilon}{2}$,

(vi)

$$T + \overline{K}_1 = \begin{bmatrix} 0 & B_1 & * & * & \cdots \\ & 0 & B_2 & * & \cdots \\ & & 0 & B_3 & \cdots \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{N}_1 \\ \mathcal{N}_2 \\ \mathcal{N}_3 \\ \mathcal{N}_4 \\ \vdots \end{matrix}$$

Since $\dim\mathcal{N}_k \le \dim\mathcal{N}_{k+1} < +\infty$, we can choose $C_k \in \mathcal{K}(\mathcal{N}_{k+1}, \mathcal{N}_k)$ with $\|C_k\| < \frac{\epsilon}{k+3}$ so that $B_k + C_k$ is surjective. Set

$$C = \begin{bmatrix} 0 & C_1 & & \\ & 0 & C_2 & \\ & & 0 & \ddots \\ & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{N}_1 \\ \mathcal{N}_2 \\ \mathcal{N}_3 \\ \vdots \end{matrix}$$

Then $C \in \mathcal{K}(\mathcal{H})$ and $\|C\| < \frac{\epsilon}{4}$. Write $\overline{K}_2 = \overline{K}_1 + C$. It is easy to see that

$$\dim(\text{Ker}(T + \overline{K}_2) \cap (\bigcap_{n\ge1} \text{Ran}(T + \overline{K}_2)^n)) = +\infty$$

and that $\bigvee_{n\geq 1} \operatorname{Ker}(T+\overline{K}_2)^n = \mathcal{H}$. Set $T+\overline{K}_2$ in the form (3):

$$T+\overline{K}_2 = \begin{bmatrix} 0 & A_1 & * & \cdots \\ & 0 & A_2 & \\ & & 0 & \ddots \\ & & & \ddots \end{bmatrix} \begin{matrix} \operatorname{Ker}(T+\overline{K}_2) \\ \operatorname{Ker}(T+\overline{K}_2)^2 \ominus \operatorname{Ker}(T+\overline{K}_2) \\ \operatorname{Ker}(T+\overline{K}_2)^3 \ominus \operatorname{Ker}(T+\overline{K}_2)^2 \\ \vdots \end{matrix}$$

Consider each A_i. By Lemmas 2.6–2.8, we can find $\overline{K}_3 \in \mathcal{K}(\mathcal{H})$ with $\|\overline{K}_3\| < \frac{\epsilon}{4}$ such that $T+\overline{K}_2+\overline{K}_3$ is strongly irreducible. Let $K = \overline{K}_2+\overline{K}_3 \in \mathcal{K}(\mathcal{H})$. Then $\|K\| < \epsilon$. This completes the proof of Theorem 2.1. □

Remark. In fact, Theorem 2.1 can be strengthened to the theorem below, and this will be useful in answering Question H.

Theorem 2.1′. *Let T be a quasitriangular operator with $\sigma(T)$ and $\sigma_w(T)$ connected. Given $\epsilon > 0$, then there exists a compact operator K with $\|K\| < \epsilon$ such that*

(i) $T+K$ is strongly irreducible,

(ii) $\sigma_p((T+K)^*) = \emptyset$,

(iii) $\operatorname{Ker}\tau_{B,T+K} = \{0\}$ if $\sigma_p(B) = \emptyset$, where $\tau_{B,T+K}$ is the Rosenblum operator.

Proof. Look back to the proof of Theorem 2.1. It is easy to see that $T+\overline{K}_2$ has dense range. We know that if $A = \{a_{ij}\}_{ij}$ is a triangular operator with respect to a suitable orthonormal basis, then $\sigma_p(A^*) \subset \{\overline{a}_{ii} : i\}$. Now we recall the proof of Lemmas 2.5–2.8.

(a) Look back to the formula (1) in the proof of Lemma 2.5. If $\operatorname{Ran}T$ is dense in $\mathcal{H}$, then $\sigma_p(T_1^*) = \sigma_p(T_2^*) = \emptyset$. To see (2), note that $\sigma_p((T+K)^*) = \emptyset$. Look back to the proof of Lemma 2.6. If $\operatorname{Ran}T$ is dense in $\mathcal{H}$, then $\operatorname{Ran}(T+K_1)$ is dense. Thus $\sigma_p((T+K)^*) = \emptyset$.

(b) Now look at (4), in the proof of Lemma 2.7. If $\overline{\operatorname{Ran}T} = \mathcal{H}$, then $\overline{\operatorname{Ran}T_3} = \mathcal{H}_2$ and $\sigma_p(T_3^*) = \emptyset$. In (5), it is clear that $\sigma_p((T_1+C_1)^*) = \sigma_p(C_2^*) = \emptyset$. So $\sigma_p((T+K)^*) = \emptyset$.

(c) Look back to (6) in the proof of Lemma 2.8. Write $W = T_{12}P_{\mathcal{L}}$ and $V = T_{12}P_{\mathcal{H}_2\ominus\mathcal{L}}$. Then $T_{12}|_{\mathcal{H}_2} = W+V$. Look at (7). Set

$$S = \begin{bmatrix} T_1 & B \\ & T_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

If $\overline{\operatorname{Ran}T} = \mathcal{H}$, then $\overline{\operatorname{Ran}T_3} = \mathcal{H}_2$. So $\sigma_p(T_3^*) = \emptyset$. If $S^*(x\oplus y) = 0$, where $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, then $T_1^*x = 0$ and $B^*x + T_3^*y = 0$. Notice that $B = C+W+V$. Since $\operatorname{Ran}W \subset \operatorname{Ran}T_1$, $W^*x = 0$. Since $\operatorname{Ran}C^* \subset \mathcal{L}$ and $\operatorname{Ran}V^* \subset \mathcal{H}_2\ominus\mathcal{L}$, it follows that $C^*x = 0$. Hence $V^*x + T_3^*y = 0$, i.e. $T^*(x\oplus y) = 0$. While $\overline{\operatorname{Ran}T} = \mathcal{H}$, $x\oplus y = 0$. Thus $\sigma_p(S^*) = \emptyset$. It is clear that $\sigma_p(T_0^*) = \emptyset$. So $\sigma_p((T+K)^*) = \emptyset$.

Summarily, in the proof of Theorem 2.1, because $\operatorname{Ran}(T+\overline{K}_2)$ is dense in $\mathcal{H}$, not only is $T+K$ strongly irreducible, but also $\sigma_p((T+K)^*) = \emptyset$.

Now we are going to prove (iii). Without loss of generality, assume that $T+K$ ($\stackrel{\text{def}}{=} A$) can be written as

$$A = \begin{bmatrix} T_1 & * & * \\ & T_2 & * \\ & & T_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{matrix}$$

where $\bigvee_{n\geq 1} \mathrm{Ker}T_i^n = \mathcal{H}_i$, $i = 1,2,3$. Suppose $BX - XA = 0$. Write $X = (X_1, X_2, X_3)$ where $X_i = X|_{\mathcal{H}_i}$. Thus $BX_1 - X_1T_1 = 0$ and hence $B^nX_1 = X_1T_1^n$ for all $n \geq 1$. If $y \in \mathrm{Ker}T_1^n$, then $B^nX_1y = 0$. By $\sigma_p(B) = \emptyset$, $X_1y = 0$. So $X_1 = 0$. Similarly, $X_2 = 0$ and $X_3 = 0$, i.e. $X = 0$. □

By the upper semi-continuity of the spectrum, the continuity of index and Theorem 2.2 of [1] or Theorem 3.49 of [3], we have the following lemma.

Lemma 2.10 ([1], [3]). *Suppose $\emptyset \neq \Gamma \subset \sigma_{le}(T)$ and $\epsilon > 0$. Then there exists a compact operator K with $\|K\| < \epsilon$ such that*

$$T + K = \begin{bmatrix} N & * \\ & \tilde{T} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

where N is a diagonal normal operator of uniform infinite multiplicity, $\sigma(N) = \sigma_{lre}(N) = \Gamma$, $\sigma(\tilde{T}) = \sigma(T)$, $\sigma_{lre}(\tilde{T}) = \sigma_{lre}(T)$, and $\mathrm{ind}(\tilde{T} - \lambda) = \mathrm{ind}(T - \lambda)$ for all $\lambda \in \rho_{S-F}(T)$.

Lemma 2.11 ([2], [3]). *Let A, B be two operators and let $\tau_{A,B}$ be the Rosenblum operator. Then the followings are equivalent:*
(i) $\sigma_r(A) \cap \sigma_l(B) = \emptyset$.
(ii) $\tau_{A,B}$ is surjective.
(iii) $\mathrm{Ran}\tau_{A,B}$ contains all compact operators.

By Corollary 2.4 of [4], it is not difficult to prove the following lemma.

Lemma 2.12 ([4], [5]). *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is quasitriangular and that $\sigma(T) = \sigma_w(T)$. Let $\Gamma = \{\lambda_n\}_{n\geq 1} \subset \sigma(T)$ satisfying the following conditions:*
(i) $\mathrm{Card}\{n : \lambda_n = \lambda_j\} = +\infty$ for all $j \geq 1$.
(ii) Each clopen of $\sigma(T)$ intersects with Γ.
Let $\epsilon > 0$ e open. Then there exists a compact operator K with $\|K\| < \epsilon$ such that $\bigvee\{\mathrm{Ker}(T + K - \lambda_n)^k : n \geq 1, k \geq 1\} = \mathcal{H}$, $\Gamma \subset \sigma_p(T + K)$ and $\sigma_p((T + K)^*) = \emptyset$.

Moreover, if $\sigma(T)$ and $\sigma_w(T)$ are connected, and if $\rho^{(n)}_{S-F}(T)$ contains a nonempty connected open subset Ω, then K can be chosen so that $T + K \in \mathcal{B}_n(\Omega)$.

Lemma 2.13 ([11]). *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose $\sigma_0(T) = \emptyset$ and $\epsilon > 0$. Then there exists a compact operator K with $\|K\| < \epsilon$ such that*
(i) $\sigma(T + K) = \sigma(T)$,
(ii) min $\mathrm{ind}(T + K - \lambda) = \begin{cases} 0, & \lambda \in \rho^{\pm}_{S-F}(T), \\ 1, & \lambda \in \rho^{(0)}_{S-F}(T) \cap \sigma(T). \end{cases}$

Lemma 2.14. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\sigma(T) \cap \rho_{S-F}(T) = \rho^{+}_{S-F}(T)$. Let $\{\Omega_j\}_j$ be the connected components of $\rho^{(1)}_{S-F}(T)$. Suppose that $\bigcup_j \Omega_j$ intersects with arbitary clopen of $\sigma(T)$. Let $\epsilon > 0$ be given. Then there exists a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ such that*
(i) $\bigvee\{\mathrm{Ker}(T + K - \lambda) : \lambda \in \bigcup_j \Omega_j\} = \mathcal{H}$ and $\sigma_p((T + K)^*) = \emptyset$.

(ii) $T+K$ has the form

$$T+K=\begin{bmatrix}\begin{bmatrix}B_1 & & & \\ ** & B_2 & & \\ ** & * & B_3 & \\ \vdots & \vdots & & \ddots\end{bmatrix} & \\ ** & B_\infty\end{bmatrix}\begin{matrix}\mathcal{M}_1\\ \mathcal{M}_2\\ \mathcal{M}_3\\ \vdots\\ \mathcal{M}_\infty\end{matrix}$$

where each B_j ($j<\infty$) is a Cowen-Douglas operator with index 1, $\sigma(B_i)\cap\sigma(B_j)=\emptyset$ ($i\neq j, i,j<+\infty$) and $\bigvee\{\mathcal{M}_j : k\leq j\leq+\infty\}$ is invariant under the commutant of $T+K$ ($1\leq k\leq+\infty$).

Proof. Let σ_j be the maximal connected closed subset of $\sigma(T)$ containing Ω_j for each j. Without loss of generality, assume that $\sigma_i\cap\sigma_j=\emptyset$ when $i\neq j$. Let Φ_j be the interior of the closure of Ω_j. Take $\alpha_j\in\Omega_j$. Let μ_j be the probability measure supported by $\partial\Phi_j$ such that $\int\varphi(z)d\mu_j(z)=\varphi(\alpha_j)$ for those functions analytic in a neighbourhood of $\overline{\Phi}_j$. Let M_j be the operator of multiplication by z on $L^2(\mu_j)$. Let $H^2(\mu_j)$ be the span of all rational functions with poles outside $\overline{\Phi}_j$. Set

$$M_j=\begin{bmatrix}M_j^+ & * \\ & M_j^-\end{bmatrix}\begin{matrix}H^2(\mu_j)\\ H^2(\mu_j)^\perp\end{matrix}$$

It is easy to show that

(1) M_j is normal and $\sigma(M_j)=\sigma_{lre}(M_j)=\partial\Phi_j$,

(2) $\sigma(M_j^-)=\overline{\Phi}_j$ and $M_j^-\in\mathcal{B}_1(\Phi_j)$.

Applying Lemma 2.10 to T^*, one can take $K_1\in\mathcal{K}(\mathcal{H})$ with $\|K_1\|<\frac{\epsilon}{2}$ such that

(3) $T+K_1=\begin{bmatrix}T_1 & * \\ & \bigoplus_j N_j\end{bmatrix}$,

(4) $\sigma(T_1)=\sigma(T)$, $\sigma_{lre}(T_1)=\sigma_{lre}(T)$ and $\text{ind}(T-\lambda)=1$ for $\lambda\in\bigcup_j\Omega_j$.

(5) N_j is diagonal normal and $\sigma(N_j)=\sigma_{lre}(N_j)=\partial\Phi_j$ for each j.

Since N_j, M_j are normal and $\sigma(N_j)=\sigma_{lre}(N_j)=\sigma_{lre}(M_j)=\sigma(M_j)$, there exists a compact operator $\overline{K}_j$ with $\|\overline{K}_j\|<\frac{\epsilon}{2^{j+3}}$ such that $N_j+\overline{K}_j\cong M_j$, where $\cong$ is the unitary equivalence relation.Thus there is a $K_2\in\mathcal{K}(\mathcal{H})$ with $\|K_2\|<\frac{\epsilon}{4}$ such that

$$\begin{aligned}T+K_1+K_2&\cong\begin{bmatrix}T_1 & * \\ & \bigoplus_j M_j\end{bmatrix}\\ &=\begin{bmatrix}T_1 & * & * \\ & \bigoplus_j M_j^+ & * \\ & & \bigoplus_j M_j^-\end{bmatrix}\\ &\stackrel{\text{def}}{=}\begin{bmatrix}T_2 & T_{12} \\ & \bigoplus_j M_j^-\end{bmatrix}\end{aligned}$$

By Theorem 3.48 of [3], choose $K_3\in\mathcal{K}(\mathcal{H})$ with $\|K_3\|<\frac{\epsilon}{8}$ such that

$$T+\sum_{j=1}^{3}K_j\cong\begin{bmatrix}T_2+C_1 & * \\ & \bigoplus_j M_j^-\end{bmatrix}$$

and $\sigma(T_2+C_1)=\sigma_w(T_2+C_1)=\sigma(T)\setminus\bigcup_j\Omega_j$. Notice that each clopen σ of $\sigma(T_2+C_1)$ intersects with the closure of $\bigcup_j\Omega_j$. There is a subset $\{\lambda_k\}_{k\geq 1}\subset\bigcup_j\Omega_j$ such that

(6) $\sup_k dist(\lambda_k,\sigma(T_2+C_1))<\dfrac{\epsilon}{16}$ and $\lim_k dist(\lambda_k,\sigma(T_2+C_1))=0$.

(7) Each clopen of $\sigma(T_2+C_1)$ contains limit points of $\{\lambda_k\}_{k\geq 1}$.

Let $\Gamma=\{\mu_k\}_{k\geq 1}$ be a dense subset of all limit points of $\{\lambda_k\}_{k\geq 1}$. By Lemma 2.12, find $K_4\in\mathcal{K}(\mathcal{H})$ with $\|K_4\|<\frac{\epsilon}{16}$ such that

$$T+\sum_{j=1}^{4}K_j\cong\begin{bmatrix}T_2+C_1+C_2 & * \\ & \bigoplus_j M_j^-\end{bmatrix}$$

where

$$T_2+C_1+C_2=\begin{bmatrix}v_1 & * & * & \cdots \\ & v_2 & * & \cdots \\ & & v_3 & \\ & & & \ddots\end{bmatrix}$$

with respect to a suitable orthonormal basis, where $v_i\in\Gamma$, and $\mathrm{Card}\{n: v_n=\mu_j\}=+\infty$ for all $j\geq 1$. Choose λ_{k_j} such that $|\lambda_{k_j}-v_j|<\frac{\epsilon}{2^5 j}$ and $\lambda_{k_j}\notin\{\lambda_{k_i}\}_{i<j}$. Perturb v_j by $\lambda_{k_j}-v_j$. Then one can find $K_5\in\mathcal{K}(\mathcal{H})$ with $\|K_5\|<\frac{\epsilon}{32}$ such that

$$T+\sum_{j=1}^{5}K_j\cong\begin{bmatrix}T_2+\sum\limits_{i=1}^{3}C_i & * \\ & \bigoplus_j M_j^-\end{bmatrix}$$

where $\sigma_0(T_2+\sum\limits_{i=1}^{3}C_i)=\{\lambda_{k_j}\}_{j\geq 1}$, $\bigvee_j \mathrm{Ker}(T_2+\sum\limits_{i=1}^{3}C_i-\lambda_{k_j})$ is equal to the acting space of $T_2+\sum\limits_{i=1}^{3}C_i$, and $\mathrm{rank}E(\lambda_{k_j},T_2+\sum\limits_{i=1}^{3}C_i)=1$. Write $\overline{T}=T_2+\sum\limits_{i=1}^{3}C_i$. Without loss of generality, assume that $\lambda_{k_j}=\lambda_j$ and that

$$T+\sum_{j=1}^{5}K_j=\begin{bmatrix}\overline{T} & * \\ & \bigoplus_j M_j^-\end{bmatrix}\begin{matrix}\mathcal{M}^\perp \\ \mathcal{M}\end{matrix}$$

Notice that $\sigma_p((T+\sum\limits_{i=1}^{5}K_i)^*)\subset\{\overline{\lambda}_j: j\geq 1\}$. If $\sigma_p((T+\sum\limits_{i=1}^{5}K_i)^*)$ is nonempty, denote it by $\{\overline{a}_j: j\}$. Write $\mathcal{H}_1=\bigvee\{\mathrm{Ker}(\overline{T}-a_j): j\}$, $\mathcal{H}_2=\mathcal{M}^\perp\ominus\mathcal{H}_1$. Then $T+\sum\limits_{j=1}^{5}K_j$ can be written as

$$T+\sum_{j=1}^{5}K_j=\begin{bmatrix}A_1 & A_{12} & A_{13} \\ & A_2 & A_{23} \\ & & \bigoplus_j M_j^-\end{bmatrix}\begin{matrix}\mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{M}\end{matrix}$$

Notice that $\bigcup_k \mathrm{Ran}(A_1-a_k)\neq\mathcal{H}_1$ and that $\bigcup_k \mathrm{Ran}(\bigoplus_j M_j^- -a_k)^*\neq\mathcal{M}$. Choose unit vectors $e\in\mathcal{H}_1\setminus\bigcup_k\mathrm{Ran}(A_1-a_k)$ and $f\in\mathcal{M}\setminus\bigcup_k\mathrm{Ran}(\bigoplus_j M_j^- -a_k)^*$. Define

$K_6x = \frac{\epsilon}{64}(x,f)e$, where (x,f) is the scalar product of x and f. Let $K = \sum_{j=1}^{6} K_j \in \mathcal{K}(\mathcal{H})$. Then $\|K\| < \epsilon$. It is an exercise to show that $\sigma_p((T+K)^*) = \emptyset$ and that $\bigvee\{\mathrm{Ker}(T+K-\lambda) : \lambda \in \bigcup_j \Omega_j\} = \mathcal{H}$. Let $\mathcal{N}_j = \bigvee\{\mathrm{Ker}(T+K-\lambda) : \lambda \in \bigcup_{i\geq j} \Omega_j\}$ and $\mathcal{M}_\infty = \bigcap_{j<\infty} \mathcal{N}_j$. Write $\mathcal{M}_j = \mathcal{N}_j \ominus \mathcal{N}_{j+1}$ for $1 \leq j < +\infty$. Then

$$T+K = \begin{bmatrix} \begin{bmatrix} B_1 & & & \\ ** & B_2 & & \\ ** & * & B_3 & \\ \vdots & \vdots & & \ddots \end{bmatrix} & \\ ** & B_\infty \end{bmatrix} \begin{matrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \vdots \\ \mathcal{M}_\infty \end{matrix}$$

It is clear that $B_j \in \mathcal{B}_1(\Omega_j)$ for $j < \infty$ and that $\sigma(B_j) \subset \sigma_j$. Furthermore, if X commutes with $T+K$, then X has the form

$$X = \begin{bmatrix} \begin{bmatrix} X_1 & & & \\ ** & X_2 & & \\ ** & * & X_3 & \\ \vdots & \vdots & & \ddots \end{bmatrix} & \\ ** & X_\infty \end{bmatrix} \begin{matrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \vdots \\ \mathcal{M}_\infty \end{matrix}$$

□

3. Proof of Main Theorem

By Theorem 2.1 and Lemma 2.13, assume that $\sigma_w(T)$ is nonconnected and

$$min\ \mathrm{ind}(T-\lambda) = \begin{cases} 0, & \lambda \in \rho^{+}_{S-F}(T), \\ 1, & \lambda \in \sigma(T) \cap \rho^{(0)}_{S-F}(T). \end{cases}$$

Suppose $\{\Omega_j\}_j$ are the connected components of $\sigma(T) \cap \rho^{(0)}_{S-F}(T)$. Let $\mathcal{H}_l = \bigvee\{\mathrm{Ker}(T-\lambda)^* : \lambda \in \bigcup_j \Omega_j\}$. Then T can be written as

$$T = \begin{bmatrix} \overline{T_1} & * \\ & \overline{T_2} \end{bmatrix} \begin{matrix} \mathcal{H}_l^\perp \\ \mathcal{H}_l \end{matrix}$$

By Lemma 2.10, choose a $\overline{K}_1 \in \mathcal{K}(\mathcal{H}_l)$ with $\|\overline{K}_1\| < \frac{\epsilon}{4}$ such that

$$\overline{T}_2 + \overline{K}_1 = \begin{bmatrix} N & * \\ & \overline{T_3} \end{bmatrix} \begin{matrix} \mathcal{H}_0 \\ \mathcal{H}_l \ominus \mathcal{H}_0 \end{matrix}$$

where N is a diagonal normal operator of uniform infinite multiplicity, $\sigma(N) = \sigma_{lre}(N) = \sigma_{lre}(\overline{T}_2)$, $\sigma(\overline{T}_3) = \sigma(\overline{T}_2)$, $\sigma_{lre}(\overline{T}_3) = \sigma_{lre}(\overline{T}_2)$ and $\mathrm{ind}(\overline{T}_3 - \lambda) = \mathrm{ind}(\overline{T}_2 - \lambda) = -1$ for $\lambda \in \bigcup_j \Omega_j$. Write $\mathcal{H}_1 = \mathcal{H}_l^\perp \oplus \mathcal{H}_0$. Set

$$K_1 = \begin{bmatrix} 0 & \\ & \overline{K_1} \end{bmatrix} \begin{matrix} \mathcal{H}_l^\perp \\ \mathcal{H}_l \end{matrix}$$

Then K_1 is compact and $\|K_1\| < \frac{\epsilon}{4}$. Look at $T + K_1$:

$$T + K_1 = \begin{bmatrix} \overline{T}_1 & * & * \\ & N & * \\ & & \overline{T}_3 \end{bmatrix} \begin{matrix} \mathcal{H}_l^{\perp} \\ \mathcal{H}_0 \\ \mathcal{H}_l \ominus \mathcal{H}_0 \end{matrix} \quad \stackrel{\text{def}}{=} \begin{bmatrix} T_1 & * \\ & \overline{T}_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_l \ominus \mathcal{H}_0 \end{matrix}$$

It is clear that $\sigma(T_1) = \sigma_w(T_1) = \sigma(T)$, $\text{ind}(T_1 - \lambda) > 0$ for $\lambda \in \rho_{S-F}(T_1) \cap \sigma(T_1)$ and $\text{ind}(T_1 - \lambda) = 1$ for $\lambda \in \bigcup_j \Omega_j$. By Lemma 2.12, take $\overline{K}_2 \in \mathcal{K}(\mathcal{H}_1)$ with $\|\overline{K}_2\| < \frac{\epsilon}{8}$ such that $A = T_1 + \overline{K}_2 \in \mathcal{B}_1(\Omega_1)$. Hence A is strongly irreducible. Notice that $\mathcal{H}_l = \bigvee\{\text{Ker}(T - \lambda)^* : \lambda \in \bigcup_j \Omega_j\}$. Each clopen of $\sigma(\overline{T}_2)$ intersects with some Ω_j. So each clopen of $\sigma(\overline{T}_3)$ contains some Ω_j. Notice that $\sigma(\overline{T}_3^*) \cap \rho_{S-F}(\overline{T}_3^*) = \rho^+_{S-F}(\overline{T}_3^*)$ and $\text{ind}(\overline{T}_3 - \lambda)^* = 1$ for $\lambda \in \bigcup_j \Omega_j$. Applying Lemma 2.14 to $\overline{T}_3^*$, find a compact operator $\overline{K}_3$ with $\|\overline{K}_3\| < \frac{\epsilon}{16}$ such that $\sigma_p(\overline{T}_3 + \overline{K}_3) = \emptyset$ and $\overline{T}_3 + \overline{K}_3$ can be written as

$$\overline{T}_3 + \overline{K}_3 = \begin{bmatrix} \begin{bmatrix} B_1 & * & * & \cdots \\ & B_2 & * & \cdots \\ & & B_3 & \\ & & & \ddots \end{bmatrix} & * \\ & B_\infty \end{bmatrix} \begin{matrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \vdots \\ \mathcal{M}_\infty \end{matrix}$$

where each B_j^* $(j < +\infty)$ is a Cowen-Douglas operator with index 1, $\sigma(B_i) \cap \sigma(B_j) = \emptyset$ when $i \neq j, i, j < +\infty$, and $\bigoplus_{i=1}^{k} \mathcal{M}_j$ is invariant under the commutant of $\overline{T}_3 + \overline{K}_3$ for each $1 \leq k \leq +\infty$. Set

$$K_2 = \begin{bmatrix} \overline{K}_2 & \\ & \overline{K}_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1 \ominus \mathcal{H}_0 \end{matrix} \in \mathcal{K}(\mathcal{H})$$

Then $\|K_2\| < \frac{\epsilon}{8}$ and

$$T + K_1 + K_2 = \begin{bmatrix} A & * \\ & \overline{T}_3 + \overline{K}_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1 \ominus \mathcal{H}_0 \end{matrix}$$
$$= \begin{bmatrix} \begin{bmatrix} A & A_{11} & A_{12} & A_{13} & \cdots \\ & B_1 & B_{12} & B_{13} & \cdots \\ & & B_2 & B_{23} & \cdots \\ & & & B_3 & \\ & & & & \ddots \end{bmatrix} & * \\ & B_\infty \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \\ \mathcal{M}_\infty \end{matrix}$$

Because $\sigma(B_i) \cap \sigma(B_j) = \emptyset$ for $i \neq j$ and $\sigma_{lre}(A) \cap \sigma_{lre}(B_j) = \sigma_{lre}(B_j) \neq \emptyset$ for all $j < +\infty$, by Lemma 2.11, we can inductively find $C_j \in \mathcal{K}(\mathcal{M}_j, \mathcal{H}_1)$ with $\|C_j\| < \epsilon/2^{j+4}$ such that $A_{11} + C_1 \notin \text{Ran}\tau_{A,B_1}$ and

$$\begin{bmatrix} A_{1,j+1} + C_{j+1} \\ B_{1,j+1} \\ \vdots \\ B_{j,j+1} \end{bmatrix} \notin \text{Ran}\tau_{A_j, B_{j+1}}$$

where

$$A_j = \begin{bmatrix} A & A_{11}+C_1 & \cdots & A_{1,j}+C_j \\ & B_1 & \cdots & B_{1,j} \\ & & \ddots & \vdots \\ & & & B_j \end{bmatrix} \begin{matrix} \mathcal{H}_l \\ \mathcal{M}_1 \\ \vdots \\ \mathcal{M}_j \end{matrix}$$

Write $D_j = A_{1,j} + C_j$. Define $K_3 x = \sum_{j<+\infty} C_j P_{\mathcal{M}_j} x$, where $P_{\mathcal{M}_j}$ is the projection onto $\mathcal{M}_j$ for each $j < +\infty$. Then $K = K_1 + K_2 + K_3$ is compact and $\|K\| < \epsilon$. Moreover,

$$T + K = \begin{bmatrix} \begin{bmatrix} A & D_1 & D_2 & \cdots \\ & B_1 & B_{12} & \cdots \\ & & B_2 & \\ & & & \ddots \end{bmatrix} & * \\ & B_\infty \end{bmatrix} \begin{matrix} \mathcal{H}_l \\ \mathcal{M}_1 \\ \mathcal{M}_2 \\ \vdots \\ \mathcal{M}_\infty \end{matrix}$$

Now we are going to prove that $T+K$ is strongly irreducible. Suppose that P is an idempotent operator commuting with $T+K$. Set

$$P = \begin{bmatrix} P_0 & Q_{01} \\ Q_{10} & \overline{P} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1^\perp \end{matrix}$$

It is easy to see that $(\overline{T}_3 + \overline{K}_3)Q_{10} = Q_{10}A$. So $(\overline{T}_3 + \overline{K}_3 - \lambda)Q_{10} = Q_{10}(A-\lambda)$ for $\lambda \in \Omega_1$. Since $A \in \mathcal{B}_1(\Omega_1)$ and $\sigma_p(\overline{T}_3 + \overline{K}_3) = \emptyset$, $Q_{10} = 0$. Furthermore, $\overline{P}$ is an idempotent operator commuting with $\overline{T}_3 + \overline{K}_3$. So $\overline{P}$ has the form

$$\overline{P} = \begin{bmatrix} \begin{bmatrix} P_1 & P_{12} & P_{13} & \cdots \\ & P_2 & P_{23} & \cdots \\ & & P_3 & \\ & & & \ddots \end{bmatrix} & * \\ & P_\infty \end{bmatrix} \begin{matrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \vdots \\ \mathcal{M}_\infty \end{matrix}$$

It is clear that P_i is an idempotent operator commuting with B_i for each $1 \le i < +\infty$, and that P_0 is idempotent and commutes with A. Since all B_j $(j < +\infty)$ and A are strongly irreducible, P_i is equal to either zero or the unit operator on its acting space for each $0 \le j < +\infty$. Set

$$P = \begin{bmatrix} \begin{bmatrix} P_0 & P_{01} & P_{02} & \cdots \\ & P_1 & P_{12} & \cdots \\ & & P_2 & \\ & & & \ddots \end{bmatrix} & * \\ & P_\infty \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{M}_1 \\ \mathcal{M}_2 \\ \vdots \\ \mathcal{M}_\infty \end{matrix}$$

If $P_0 = 0$, then $AP_{01} + D_1 P_1 = P_{01} B_1$. Since $D_1 \notin \mathrm{Ran}\tau_{A,B_1}$, $P_1 = 0$. It follows from $P^2 = P$ that $P_{01} = 0$. Similarly, one can inductively prove that $P_j = 0$ for $j < +\infty$ and that $P_{i,j} = 0$ for $i, j < \infty$. Thus

$$\overline{P} = \begin{bmatrix} 0 & * \\ & P_\infty \end{bmatrix} \begin{matrix} \mathcal{H}_1^\perp \ominus \mathcal{M}_\infty \\ \mathcal{M}_\infty \end{matrix}$$

It is clear that $\mathrm{Ran}\overline{P}^* \cap \mathrm{Ker}(\overline{T}_3 + \overline{K}_3 - \lambda)^* = \{0\}$ for all $\lambda \in \bigcup_j \Omega_j$. So $\overline{P} = 0$. Hence $P = 0$. If P_0 is the unit operator acting on $\mathcal{H}_1$, then one can show that $P = I$. So $T + K$ is strongly irreducible.

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